taken from the fixed point (x_0, y_0) to the point (x, y) over any continuous curve formed of a finite number of arcs of the type (2) defines a single-valued function $\phi(x, y)$. For any two broken curves joining (x_0, y_0) with (x, y) form a polygon over which the value of the integral is zero, and the two values found by integrating from (x_0, y_0) to (x, y) over the two broken curves, are equal on account of the property (4).

The difference of two values of $\phi(x, y)$ corresponding to points on the same ordinate has the value

$$\phi(x, y) - \phi(x, y_1) = \int_{y_1}^{y} B(x, y)\,dy. \tag{5}$$

This difference can in fact be written in the form

$$\phi(x, y) - \phi(x, y_1) = \int_{(x-h, y_1)}^{(x, y)} (A + By')\,dx - \int_{x-h}^{x} A(x, y_1)\,dx, \tag{6}$$

since from the figure it is evident that

$$\phi(x, y) = I(C) + \int_{(x-h, y_1)}^{(x, y)} (A + By')\,dx,$$

$$\phi(x, y_1) = I(C) + \int_{x-h}^{x} A(x, y_1)\,dx.$$

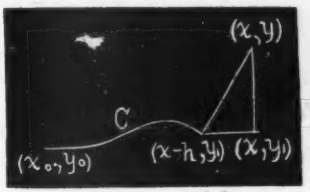

Fig. 2.

Equation (6) may also be written, by a change of variable,

$$\phi(x, y) - \phi(x, y_1) = \int_{(x-h, y_1)}^{(x, y)} A\,dx + \int_{y_1}^{y} B\,dy - \int_{x-h}^{x} A(x, y_1)\,dx.$$

When h approaches zero the first and third of these integrals vanish, and the second approaches the value given in equation (5).

The derivatives of $\phi(x, y)$ can now be readily calculated. From (5) it follows that

$$\frac{\partial \phi}{\partial y} = B(x, y).$$

It is evident from a figure similar to figure 2 that

$$\phi(x, y) - \phi(x_1, y) = \int_{x_1}^{x} A(x, y)\,dx,$$

so that

$$\frac{\partial \phi}{\partial x}=A(x, y).$$

A necessary and sufficient condition for the integral I to be independent of the path in a region R is therefore that a single-valued function $\phi(x, y)$ exists in R having the derivatives

$$\frac{\partial \phi}{\partial x}=A, \quad \frac{\partial \phi}{\partial y}=B. \tag{7}$$

The sufficiency of this condition was not proved above, but follows easily with the help of the fundamental theorem of the integral calculus. For along any arc (2)

$$\int_{(x_1, y_1)}^{(x_2, y_2)}(A+By')dx=\int_{(x_1, y_1)}^{(x_2, y_2)}\left(\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}y'\right)dx=\phi(x_2, y_2)-\phi(x_1, y_1).$$

Another criterion for the invariancy of I is the following:

If the functions A and B in the integral (1) *have continuous partial derivatives $\frac{\partial A}{\partial y}$ and $\frac{\partial B}{\partial x}$ in a simply connected region R*, then a necessary and sufficient condition for the integral I to be independent of the path, is that the equation*

$$\frac{\partial A}{\partial y}=\frac{\partial B}{\partial x} \tag{8}$$

is identically satisfied in R.

The condition is necessary, for from the previous theorem a function ϕ must exist with the derivatives (7), and we have

$$\frac{\partial A}{\partial y}-\frac{\partial B}{\partial x}=\frac{\partial^2 \phi}{\partial x \partial y}-\frac{\partial^2 \phi}{\partial y \partial x}=0.$$

To prove the sufficiency, suppose first that R is a rectangle with one corner at (x_0, y_0). Then the values of I taken from (x_0, y_0) to (x, y) along lines parallel to the x and y axes define a function

$$\phi(x, y)=\int_{x_0}^{x}A(x, y_0)dx+\int_{y_0}^{y}B(x, y)dy, \tag{9}$$

*R is said to be simply connected if any two of its points can be connected by a continuous curve, and if the interior of any continuous, closed, non-intersecting curve in R is also entirely within the region.

which is readily seen with the help of equation (8) to have the partial derivatives A and B. Consequently in any rectangle where condition (8) is satisfied, the integral I is independent of the path.

In a more general simply-connected region R, the value of I taken around any polygon whose sides are parallel to the x- or y-axis is zero. For by continuing all the sides of the polygon, its interior is divided into rectangles, and the sum of the values of I taken around these rectangles in the positive direction* is the value of I taken in the positive direction about the original polygon. This follows because we have to integrate along a side of any rectangle twice, in opposite directions, unless the side is an edge of the original polygon. But since R is simply connected, the rectangles are all in R when condition (8) holds, and the value of I taken around the edge of such a rectangle is zero. Hence if we integrate from a fixed point (x_0, y_0) to the point (x, y) in R along a broken curve consisting of straight lines parallel to one or the other of the axes, a single-valued function $\phi(x, y)$ is defined. This function like the function defined by equation (9) has A and B for its partial derivatives.

For the application to be made in the next section, the following remark is important.

If the integral (1) *takes the same value over all continuous curves consisting of a finite number of arcs of the type* (2) *joining two fixed points* (x_1, y_1) *and* (x_2, y_2), *then it must be independent of the path in the same way for any two points in the strip of the plane between the ordinates* $x=x_1$ *and* $x=x_2$.

Let 3 and 4 be any two points in this strip, and join 1 with 3, and 4 with 2 by fixed arcs of the type (2). Then however the points 3 and 4 are connected by broken arcs, we will always, by hypothesis, have the same value for

$$I_{13}+I_{34}+I_{42}.$$

Consequently, I_{34} is independent of the path also.

§3. The Derivation of Euler's Equation.

The problem of the calculus of variations which we shall consider here is the problem of finding a curve which joins two given fixed points (x_0, y_0) and (x_1, y_1), and gives a maximum or a minimum value to an integral of the form

$$J=\int f(x, y, y')dx.$$

The function f under the integral sign will be supposed to have continuous derivatives of the first and second orders for points (x, y) in a region R of

*I. e. keeping the interior of the rectangle on the left.

the xy-plane and all values of y'. Evidently J will then have a well-defined value over any continuous curve consisting of a finite number of arcs of the type (2) in R.

Suppose that one of these arcs

$$E; \qquad y=y(x), \quad x_0 \leqq x \leqq x_1, \tag{10}$$

joins the points 0 and 1 in R and gives J a minimum value. The family of arcs

$$v; \qquad \bar{y}=y(x)+\alpha\eta(x), \quad x_0 \leqq x \leqq x_1,$$

where α is an arbitrary constant and $\eta(x)$ satisfies the conditions

$$\eta(x_0)=\eta(x_1)=0,$$

also joins the points 0 and 1 and include E for the value $\alpha=0$. If the function $\eta(x)$ has a derivative which is continuous except perhaps at a finite number of points in the interval $x_0 \leqq x \leqq x_1$, then along any arc v the integral J will have a value

$$J(\alpha)=\int_{x_0}^{x_1} f(x, \bar{y}, \bar{y}')\,dx,$$

which is a function of α. Since v reduces to E when $\alpha=0$, it follows that for $\alpha=0$ the function $J(\alpha)$ must have a minimum and $\frac{dJ}{d\alpha}$ must be zero.

The derivative $\frac{dI}{d\alpha}$ for $\alpha=0$ is easily found to be

$$\left(\frac{dJ}{d\alpha}\right)_{\alpha=0}=\int_{x_0}^{x_1}\left(\frac{\partial f}{\partial y}\eta+\frac{\partial f}{\partial y'}\eta'\right)dx, \tag{11}$$

where in the derivatives of f the values of y and y' along E are substituted. This derivative must vanish however the function $\eta(x)$ satisfying the conditions given above, is chosen. We see then that in the $x\eta$-plane the integral (11) is independent of the path for all curves joining the points $(x_0, 0)$ and $(x_1, 0)$, and consequently independent of the path anywhere in the strip of the $x\eta$-plane between the ordinates $x=x_0$ and $x=x_1$. There must therefore exist a function $\phi(x, \eta)$ which has the derivatives

$$\frac{\partial\phi}{\partial x}=\frac{\partial f}{\partial y}\eta, \quad \frac{\partial\phi}{\partial\eta}=\frac{\partial}{\partial y'}. \tag{12}$$

From the first of these equations,

$$\phi=\eta\int_{x_0}^{x}\frac{\partial f}{\partial y}dx+H(\eta), \tag{13}$$

and from the second,

$$\frac{\partial f}{\partial y'}=\int_{x_0}^{x}\frac{\partial f}{\partial y}dx+H', \tag{14}$$

an equation which must be true at every point of E. It is evident by deriving again for η, that H'' must be zero, and H has the form

$$H=c\,\eta+d. \tag{15}$$

If the curve E has corners for the values $x=a, a_2, \ldots, a_n$ in the interval $x_0 \leqq x \leqq x_1$, then the functions (12) may have discontinuities, but they will certainly be continuous in any strip of the $x\,\eta$-plane between two ordinates $x=a_k$, $x=a_{k+1}$. In such a strip ϕ must therefore be continuous and it is evident from equations (13) and (15) that c has the same value throughout the strip.

In two different strips c is also the same. For simplicity consider the case when there is but one value $x=a$, and let

$$\phi=\eta\int_{x_0}^{x}\frac{\partial f}{\partial y}dx+c\,\eta+d, \quad \psi=\eta\int_{x_0}^{x}\frac{\partial f}{\partial y}dx+e\,\eta+f \tag{16}$$

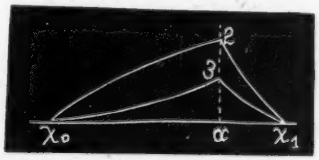

Fig. 3.

be the two functions (13) in the two strips. Along two curves such as those in figure 3,

$$I=\phi_2-\phi_0+\psi_1-\psi_0=\phi_3-\phi_0+\phi_1-\phi_3,$$

the subscripts indicating the points at which the values of the functions are to be taken. Hence

$$\phi_2-\phi_3=\psi_2-\psi_3,$$

from which it follows with the help of equations (16) that $c=e$.

From equation (14), therefore,

$$\frac{\partial f}{\partial y'}=\int_{x_0}^{x}\frac{\partial f}{\partial y}dx+c, \tag{17}$$

where c has the same value throughout the whole interval $x_0 \leqq x \leqq x_1$.

Several important conclusions can be drawn from equation (17). In the first place consider a point 2 on E when y' is continuous. The left member of (17) may be regarded for the moment as a function of the two varibles y' and x, the latter entering explicitly and also in the function $y(x)$. Then the values x_2, y_2, y'_2 at the point 2 on E furnish a solution of equation (17). If $\frac{\partial^2 f}{\partial y'^2}$ is different from zero for these values, then the theory of implicit functions tells us that equation (17) has but one continuous solution $y'(x)$ which reduces to y'^2 when $x=x_2$, and this solution has a continuous derivative. The values of $y'(x)$ on the curve E near the point 2 constitute this solution, and by differentiating equation (17) for x we derive the following theorem:

If an arc E

$$y=y(x), \quad x_0 \leqq x \leqq x_1,$$

minimizes the integral J, then at any point on E where $y'(x)$ is continuous and $\frac{\partial^2 f}{\partial y'^2}$ different from zero, the function $y(x)$ must also have a second derivative and satisfy the Euler differential equation

$$\frac{d}{dx}\frac{\partial f}{\partial y'}-\frac{\partial f}{\partial y}=0.$$

It is possible to show further from equation (17) that a minimizing arc can not in general have corner points. At a point (x_2, y_2) of E where $\frac{\partial^2 f}{\partial y'^2}$ is different from zero for all values of y', it is evident that the first member of equation (17) can equal the second for at most one value of y', since $\frac{\partial f}{\partial y'}$ is monotomic. Hence it would be impossible for y' to be discontinuous at (x_2, y_2).

An arc E which minimizes the integral J can not have a corner point at any point (x_2, y_2) where $\frac{\partial^2 f}{\partial y'^2}$, is different from zero for all values of y'. If $\frac{\partial^2 f}{\partial y'^2}$ is different from zero at any point in R for all y''s, then no minimizing curve whatsoever with corner points is possible anywhere in R.

THE PRYTZ PLANIMETER.

By DR. A. R. CRATHORNE, University of Illinois.

The object of this article is to call attention to a little known but very interesting and simple mathematical instrument, the Prytz planimeter, or to give its more common name—the hatchet planimeter. It is easily made from a piece of stiff wire which is bent into the form shown in Fig. 1. One end of the wire is ground to a point and the other end is flattened into a chisel edge. The sharp point and this edge should be in the same plane. For a given area a planimeter which is longer than the longest diameter should be used.

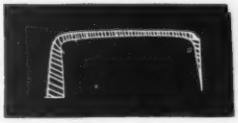

Fig. 1.

To obtain the area of a closed curve with this planimeter a straight line of indefinite length is drawn from the approximate center of gravity in any direction (Fig. 2). The pointed end of the instrument is placed on the center of gravity and the chisel edge on the straight line (at A in the figure). The legs of the planimeter must at all times be perpendicular to the plane of the area. From its position B, the tracing point is now moved along the straight line to the intersection with the boundary curve and then around the area in the direction indicated and back to B. The edged end of the instrument, upon which a slight pressure is brought to bear, traces out the curve $ADEFHIK$ and when the tracing point has returned to B, will take the position K. The product of the length AK into the length of the planimeter will be the approximate area of the given curve (see equation [7]).

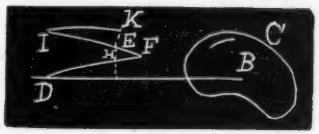

Fig. 2.

This instrument was invented some fifteen years ago by Captain Prytz of the Danish Army, who published an account of his invention in the English magazine, *Engineering*, Vol. 72, page 813. A detailed analytical discussion of its theory was given by M. F. W. Hill in the *Philosophical Magazine* for 1894. In the *Bulletin de l' Académie Imperiale des Sciences de St. Petersbourg*, 1903, Professor Kriloff of the Russian Naval Academy discussed the instrument from the geometrical standpoint and gave a very elementary and simple explanation of its theory.

This theory depends on the well known theorem:* The total area Z swept out by a straight line AB moving in a plane is given by the formula

$$[1] \qquad Z = ls + (\tfrac{1}{2}l^2 - al)(\theta_2 - \theta_1),$$

*See Chapter XIV, Gibson's *Calculus*.

where l is the length of the moving line; s, the total normal displacement of a point P of the line; a, the distance AP; and θ_1, θ_2 are the initial and final values of θ, the angle made by the moving line with some fixed line. If, in particular, the point A moves around a closed curve C', while at the same time B makes a complete circuit of the curve C which lies entirely outside of C', then $\theta_2=\theta_1$, and we have

[2] $$Z=C-C'=ls.$$

The usual conventions as to the signs of the areas in question hold. Areas covered twice in opposite directions by the line are zero, and an area described counter clock wise is positive. If a small wheel with its axis in the moving line be attached at P, the arc through which it turns will give us the normal displacement s. We will call this wheel the measuring wheel. It should be noticed that the position of the point P does not enter into equation [2]. In measuring areas with most planimeters the area of the curve C' is a constant of the instrument. In the case where the point corresponding to A moves backwards and forwards on a curve this constant is zero. In the well known Amsler polar planimeter the curve C' is a circle, and in some other instruments it is a straight line. The area C is the area to be measured.

In the Prytz planimenter there is no fixed curve C', but instead we have a curve which depends upon the curve whose area we are measuring. Referring to figure 2, we see that as the point B moves in its path, the edge A moves in its curve of pursuit from A to K. Now turn the planimeter horizontally about the point B until it is in the initial position AB. The curve corresponding to the curve C' is now a closed curve $ADEFHIKA$. The total area swept out by the line AB is equal to the algebraic sum of the area C and the areas ADE, EFH, and HIK. The other parts of the plane swept by the line are covered twice in cpposite directions and hence do not enter into the algebraic sum. Putting in the proper signs we have

[3] $$\text{Total area swept out}=C-ADE+EFH-HIK.$$

From equation [2] the total area swept out is measured by the product of the length l of AB into the length of arc s through which a measuring wheel at A on the line AB would have turned. During the motion of the tracing point this wheel does not turn at all, for the direction of motion is perpendicular to the edge of the wheel. But in turning the instrument about B from the position KB into AB, this wheel will turn through an arc which is equal in length to $l\phi$ where ϕ is the angle KBA. Or we have

[4] $$s=l\phi.$$

The total area swept out by AB is ls or $l^2\phi$. This gives the exact equation,

[5] $$l^2\phi = C - ADE + EFH - HIK.$$

If the starting point for tracing the figure be taken as above (*i. e.* at the center of gravity of the area), the algebraic sum of the three areas enclosed by the curve of pursuit and the arc KA will be very nearly zero and we have the approximate equation

[6] $$c = l^2\phi = l.\text{arc}\,KA.$$

If the angle ϕ be small, say less than 20°, the arc KA can be replaced by its chord, and we have

[7] $$c = l.KA.$$

If an area whose longest diameter is four inches or less, be measured with a ten inch planimeter, the error is very small and is about equal to the error made in finding the area of an equivalent rectangle by measuring the sides with a scale. The error due to the non-alignment of the edge and tracing point can be eliminated by tracing the curve in opposite directions and finding the mean of the two results.

An improved planimeter of this type has a small chisel-edged wheel instead of the chisel edge.

EXISTENCE OF A MINIMUM OF A QUADRATIC FUNCTION.

By T. H. HILDEBRANDT, The University of Chicago.

Suppose we have a quadratic function in n variables,

$$F[x_1, \ldots, x_n] = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i\, x_j + \sum_{i=1}^{n} b_i\, x_i + c,$$

of which we know that

$$F[x_1, \ldots, x_n] \geqq 0,$$

for all values of the variables $x_1, \ldots, x_n$. For instance, such a function is

$$\int_0^{2\pi} [\phi(x) - \sum_{i=1}^{n} (a_i \cos ix + b_i \sin ix)]^2\, dx,$$

where the a_i and b_i take the place of x_i. It might appear that because the function has a lower bound, it also has a minimum, *i. e.*, reaches its greatest lower bound. But this statement is not sufficient to establish the fact that it reaches its greatest lower bound in a finite point. If the x_i were restricted to be finite, we could apply the theorem on continuous functions of n variables, which states that in such a case the function has a minimum. But the theorem does not apply when the x_i are permitted to assume infinite values also. We proceed to show that even in this case there is at least one point in the finite part of n-space at which F takes on its minimum value.

Transform F by a linear homogeneous substitution:

$$x_i = \sum_{j=1}^{n} a_{ij}\, x'_i\, x'_j \qquad i=1, \ldots, n,$$

of such a nature that the determinant $| a_{ij} |$ of the substitution is different from zero and so that in the transformed function f the cross product terms $x'_i\, x'_j$ disappear. This is always possible, as a matter of fact, even in the special case where the substitution is orthogonal and therefore $| a_{ij} | = 1$. Our function then takes the form:

$$\sum_{i=1}^{n} a'_i\, x'^2_i + \sum_{i=1}^{n} b'_i\, x'_i + c = f(x'_1, \ldots, x'_n),$$

while $f(x'_1, \ldots, x'_n) = F(x_1, \ldots, x_n) \geq 0$. Then we have $a'_i \geq 0 \quad (i=1, \ldots, n)$. For, suppose $a_i' < 0$. Then let $x'_1, \ldots, x'_{j-1}\, x'_{j+1}, \ldots, x'_n$ take the zero values. We have

$$f(0, \ldots, 0, x_j, 0, \ldots, 0) = a_j'\, x_j^2 + b_j\, x_j + c.$$

But evidently since $a'_j < 0$ we may choose x'_j so large that this becomes negative, contrary to the hypothesis that f shall be greater than or equal to zero for all values of the x'_j. Moreover, if $a'_j = 0$, then $b'_j = 0$. Suppose $a'_j = 0$ and $b'_j < 0$. Then by taking for $x'_1, \ldots, x'_{j-1}, x'_{j+1}, \ldots, x'_n$, the values zero, we can find a sufficiently large positive value for x'_j to make $b'_j x'_j + c'$ negative, and hence, f negative. Hence, if $a'_j = 0$, then $b'_j = 0$. Our function is then of the form:

$$f(x'_1, \ldots, x_r') \equiv \sum_{i-1}^{r} [a'_i\, x'^2_i + b'_i\, x'_i + c] \qquad (r \leqq n).$$

Transform this function by the substitution

$$x_i'' = x'_i + \frac{b_i}{2a_i}.$$

Then we obtain

$$F(x_1, \ldots, x_n) \equiv \phi(x_1'', \ldots, x_n'') \equiv \sum_{i=1}^{r} a_i' x_i''^2 + c' \geqq 0.$$

Evidently $c' \geqq 0$. Moreover we now see at once the minimum value of ϕ, namely,

$$x_1''=0, \ldots, x_n''=0.$$

By putting $x_i''=0$ and solving our transformations backward we can obtain the values of $x_1, \ldots, x_n$ which will make $F(x_1, \ldots, x_n)$ a minimum. There will be a unique solution if $r=n$. For, then we will have n equations in n unknowns. If $r<n$, the solution is not unique, but there will be at least a single infinity of values which will make F a minimum. When $r=n$, our function f evidently has the form:

$$f(x'_1, \ldots, x'_n) = \sum_{i=1}^{n} (a'_i x'^2_i + b'_i x'_i + c).$$

Now $\sum_{i=1}^{n} a'_i x'^2_i$ is a positive, definite form, and so $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$ must also be. That is, the condition that

$$F(x_1, \ldots, x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j + \sum_{i=1}^{n} b_i x_i + c \geqq 0$$

may have a minimum is that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$$

be a positive definite form. This is exactly the result obtained by applying the methods of the differential calculus.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

253. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that $x^5+ax+b=0$ is solvable by radicals if $b=ma$, m being the negative of half the sum of any two roots of the original equation. Exhibit the solution.

Solution by DR. L. E. DICKSON, The University of Chicago.

Let r and s be the two roots for which $m=-\frac{1}{2}(r+s)$. Then

$$r^5+ar+ma=0,\ s^5+as+ma=0.$$

Adding, we get $r^5+s^5=0$. We exclude the trivial case $m=0$, so that $r+s\neq 0$, $r=-es$, e being an imaginary fifth root of unity. From $r+es=0$, $r+s=-2m$, we get

$$(1)\qquad s=\frac{2m}{e-1},\ r=\frac{-2me}{e-1}.$$

Now s will indeed be a root if and only if

$$32m^4+al=0,\ l=(e+1)(e-1)^4.$$

Since $e^4+e^3+e^2+e+1=0$, we get $l=5(e^3+e^2+1)$. Now e^2+e^3 and $e+e^4$ are the roots $\frac{1}{2}(-1\pm\sqrt{5})$ of $z^2+z-1=0$. Hence the problem stated is possible if and only if

$$64m^4+5(1\pm\sqrt{5})a=0,\ a=-\tfrac{16}{5}(1\mp\sqrt{5})m^4.$$

For m arbitrary, and for this value of a, the equation $x^5+ax+ma=0$ has the two roots (1); the remaining roots may be found by solving a cubic.

284. Proposed by DR. E. H. MOORE, The University of Chicago, Chicago, Ill.

Discuss the system of equations:

$$\begin{cases} x^k+y^k=a_k \\ x^l+y^l=a_l \end{cases}\quad (k,\ l \text{ distinct positive integers})$$

in general and for particular values of $(k,\ l;\ a_k,\ a_l)$.

Solution by BENJAMIN F. FINKEL, Ph. D., Drury College, Springfield, Mo.

§1. General Discussion.

Assume $k<l$. Solve the first equation for y, say, and we have $y=(a_k-x^k)^{1/k}$. Substitute this value of y in the second equation and free the resulting equation from fractional exponents. The equation thus obtained is called the *Resultant Equation** in x, and is, at most, of the klth degree. It has, therefore, kl roots, real, imaginary, or infinite.

If k and l are both odd or both even, the resultant equation is of the $kl-k$th degree and there are kl roots, k of which are infinite, and, therefore, $kl-k$ roots, real or imaginary.

If k is odd and l even, or if k is even and l odd, the resultant equation is of the klth degree and there are kl roots.

Since the equations are symmetric in x and y the resultant equation for y is the same as that for x. Hence x and y have the same series of values, but not every value of y can be taken with every value of x and satisfy the given system of equations. The final test of the legitimacy of every value of x and y is that these values when substituted in the given system of equations shall satisfy them, and every such set of values of x and y constitutes a solution.

Since the two equations of the system are symmetric with respect to x and y, it follows that if $(x, y)\equiv(a, b)$, $a\neq b$, is a solution, then $(x, y)\equiv(b, a)$ is also a solution. Since, as we have seen above, the greatest number of sets of values of x and y satisfying the given equations is kl, it follows that these sets are made up of sets of values of x and y in which x and y have different values, the number of such sets not exceeding $\frac{1}{2}kl$, and sets of values of x and y in which x and y have the same value.

Let us assume that there are kl solutions of the given system of equations and let $2n'$ be the number of sets of values of x and y wherein x and y have different values, *e. g.*, $(x,y)\equiv(a, b)$, $a\neq b$, and let n'' be the number of sets wherein x and y have the same values, *e. g.*, $(x, y)\equiv(c, c)$.

Then we must have

$$2n'+n''=kl.$$

kl is of the form $2m$ or $2m+1$.

(1) Assume kl is of the form $2m$. Then we have $2n'+n''=2m$, the solutions of which are

n'	n''
m	0
$m-1$	2
$m-2$	4
⋮	⋮
0	$2m$

*Chrystal's *Algebra*, Part I, p. 403, ed. 1885.

(2) Assume kl of the form $2m+1$. Then we have $2n'+n''=2m+1$, the solutions of which are

n'	n''
m	1
$m-1$	3
$m-2$	5
⋮	⋮
0	$2m+1$

Hence, it follows that when kl is even the number of solutions in which x and y are equal must be even, and when kl is odd, the number of such solutions must be odd.

The graph of the equation $x^n+y^n=a_n$, when n is even and greater than 2 and $a_n>0$ is of the form of Fig. 1. When $n=2$ and $a_n>0$, the graph is a circle.

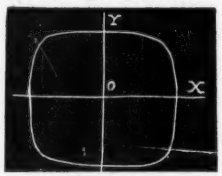

Fig. 1.

When n approaches an infinitely large even number and a_n is a finite positive number, the graph approaches a square whose sides are 2. When $a_n<0$, the graph is imaginary.

The graph of the equation $x^n+y^n=a_n$, when n is odd and greater than 1 is of the form of Fig. 2. When n is 1, the graph is a straight line. When n approaches an infinitely large positive odd integer and a_n is a finite positive number, the graph approaches the full-line curve in Fig. 3. When n approaches an infinitely large odd integer and $a_n<0$, the graph approaches the dotted line in Fig. 3.

Fig. 2.

If, instead of solving one of the equations with respect to one of the unknowns and substituting this value in the second, to obtain the resultant equation in the other unknown, one solves the two equations for the ratio, r, of x to y, there results, in general, an equation of the klth degree in r. And since the two equations are symmetric in x and y, this equation in r will be a *reciprocal equation.* This fact often enables one to obtain solutions more easily than by the first method. It also enables one to pair the proper values of x and y. It does not, however, prevent the introduction of extraneous roots. Thus, after r is found, and one wishes to find x, say, there will be for each of the kl values of r as many values of x as there are units in the exponent of x in the equation in which r is substituted. Thus, suppose $x^k(1+r^k)=a_k$; from this equation one would obtain k values of x for each of the kl values of r. One would thus obtain k^2l values in all for x, of which not more than kl can be legitimate values, that is, values which when used with certain values of y will satisfy the system of equations.

Fig. 3.

§2. Special Cases.

a. If $k=l$ and $a_k=a_l$, then there are an infinitude of solutions. Geometrically interpreted, the two equations represent two coincident curves. Hence, the co-ordinates of all points of the curves will satisfy the two equations.

If $k=l$ and $a_k \neq a_l$, the roots are all infinite (real or imaginary). Thus, let $k=l=1$ and, therefore, $x+y=a$ and $x+y=b$, $a \neq b$. These equations are frequently called inconsistent, but for the sake of uniformity of language they are commonly said to be satisfied by the point at infinity. $(x, y) \equiv (\infty, -\infty)$, $(-\infty, \infty)$ are solutions. It thus appears that there are two solutions here. We may, however, consider this to be really only one solution; for geometrical considerations have led us to regard $-\infty$ and ∞ as the same point if the real number system be represented by points in a straight line.* The equations represent two parallel straight lines.

Suppose $x^2+y^2=a^2$ and $x^2+y^2=b^2$, $a \neq b$. Here, $x=\pm\infty$ and hence $y=\pm\infty i$. Since x and y are symmetric we have 2^3 solutions. These are, $(\infty, \infty i)$, $(\infty, -\infty i)$, $(-\infty, \infty i)$, $(-\infty, -\infty i)$, $(\infty i, \infty)$, $(\infty i, -\infty)$, $(-\infty i, \infty)$, and $(-\infty i, -\infty)$. But from the previous considerations, these eight solutions may be reduced to four, the requisite number; for we may consider $(\infty, \infty i)$, $(\infty, -\infty i)$ as one point, and $(-\infty, \infty i)$, $(-\infty, -\infty i)$ as a second point and so on. If we assume that parallel lines in space intersect at infinity, these four solutions may be reduced to two solutions, the first four above constituting one solution and the last four a second solution.

By a more generally accepted convention, we may consider the two solutions as the "circular points at infinity," through which two points every circle passes. The co-ordinates of one of the circular points at infinity may be considered to be $(\infty, \infty i)$ or $(-\infty, -\infty i)$; these co-ordinates, by the previous consideration, being one and the same point, and the co-ordinates of the other point at infinity being $(\infty, -\infty i)$ or $(-\infty, \infty i)$. These points may be constructed geometrically by the method suggested in Carr's *Synopsis of Pure and Applied Mathematics*, pp. 674-677.

Suppose $x^3+y^3=a$, $x^3+y^3=b$, $a \neq b$. Here, $x=\infty, \omega\infty, \omega^2\infty$, and $y=-\infty, -\omega\infty, -\omega^2\infty$. There are here eighteen algebraic solutions, but these, as before, may be reduced to nine, the requisite number.

Suppose $x^4+y^4=a$, $x^4+y^4=b$, $a \neq b$. Here, $x=\pm\infty, \pm\infty i$, and $y=\pm\infty \sqrt{i}, \pm\infty i^3$. There are here sixteen solutions, the requisite number.

b. Suppose $k=1$, $l=2$, and, therefore, $kl=2$. Here, $2n'+n''=2$. There are two possibilities:

n'	n''
1	0
0	2

*Reye, *Geometrie der Lage*, p. 18.

As an example of the first possibility, let the system be

$$\begin{cases} x+y=3, \\ x^2+y^2=5, \end{cases} \quad (x, y) \equiv (1, 2), (2, 1).$$

In order that the second possibility be satisfied, a_1^2 must equal $2a_2$. As an example of the second possibility, let the system be

$$\begin{cases} x+y=6, \\ x^2+y^2=18, \end{cases} \quad (x, y) \equiv (3, 3), (3, 3).$$

c. $k=1$, $l=3$, and, therefore, $kl=3$. Here, $2n'+n''=3$. There are two possibilities:

n'	n''
1	1
0	3

If we solve the system, $\begin{cases} x+y=a_1, \\ x^3+y^3=a_3, \end{cases}$ we find $(x, y) \equiv (\infty, \infty)$,

$$\left(\frac{a_1}{2} \pm \tfrac{1}{2}\sqrt{\frac{4a_3-a_1^3}{3a_1}}, \quad \frac{a_1}{2} \mp \tfrac{1}{2}\sqrt{\frac{4a_3-a_1^3}{3a_1}}\right).$$

(1) If $a_1=3$, $a_3=9$; $(x, y) \equiv (\infty, \infty)$, $(1, 2)$, $(2, 1)$, an example of the first possibility. The graph of this system is Fig. 4.

(2) In order that the second possibility be satisfied, a_1^3 must equal $4a_3$ or $a_1=0$.

Let $a_1=6$, and $a_3=54$. Then $(x, y) \equiv (\infty, \infty)$, $(3, 3)$, $(3, 3)$. The graph of the system is Fig. 5.

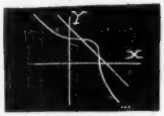

Fig. 4.

d. $k=1$, $l=4$, and, therefore, $kl=4$. Here, $2n'+n''=4$. There are three possibilities:

Fig. 5

n'	n''
2	0
1	2
0	4

Solving the system of equations $x+y=a_1$, $x^4+y^4=a_4$, we find (x, y)

$$\equiv \left(\frac{a_1 \pm \sqrt{[-3a_1^2 \mp 2\sqrt{(2a_1^4+2a_4)}]}}{2}, \quad \frac{a_1 \pm \sqrt{[-3a_1^2 \pm 2\sqrt{(2a_1^4+2a_4)}]}}{2}\right).$$

(1) If $a_1=3$, $a_4=17$; $(x, y)\equiv(1, 2)$, $(2, 1)$, $(3\pm\sqrt{-55}, 3\mp\sqrt{-55})$, an example of the first possibility. The graph of the system is the Fig. 6.

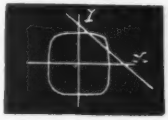

Fig. 6.

(2) For the second possibility, a_1^4 must equal $8a_4$.

If $a_1=4$ and $a_4=32$, $(x, y)\equiv(2,2)$, $(2,2)$, $(2\pm2\sqrt{-6}, 2\mp2\sqrt{-6})$.

a_1 and a_4 must be so taken that $2a_1^4+2a_4$ is a perfect square. Thus, if $a_1=1$ and $a_4=8$, we have an example of the first possibility. If $a_1^4=-a_4$, we have an example also satisfying the first possibility, but since the graph of $x^4+y^4=-1$ is imaginary, all the roots are imaginary.

(3) The only example satisfying the third possibility is $x+y=0$, and $x^4+y^4=0$. Here, $(x, y)\equiv(0, 0)$, $(0, 0)$, $(0, 0)$, $(0, 0)$.

e. $k=1$, $l=5$, and, therefore, $kl=5$. Here, $2n'+n''=5$. There are three possibilities:

n'	n''
2	1
1	3
0	5

Solving the system of equations, $x+y=a_1$, $x^5+y^5=a_5$, we find $(x, y)\equiv(\infty, \infty)$,

$$\left[\tfrac{1}{2}\left(a_1 \pm\sqrt{-a_1^2\mp\tfrac{1}{5}\sqrt{[(5a_1^5+20a_5)/a_1)}}\right),\right.$$

$$\left.\tfrac{1}{2}\left((a_1\pm\sqrt{-5a_1^2\mp\tfrac{2}{5}\sqrt{[(5a_1^5+20a_5)/a_1]}}\right)\right].$$

(1) If $a=3$ and $a_5=33$, $(x, y)\equiv(1, 2)$, $(2, 1)$, $[\frac{1}{2}(3\pm\sqrt{-19}), \frac{1}{2}(3\mp\sqrt{-19})]$, (∞, ∞).

Fig. 7.

(2) For the second possibility, a_1^5 must equal $16a_5$, and $(5a_1^5+20a_5)/a_1$ must be a perfect square. If $a_1=4$ and $a_5=64$, $(x, y)\equiv(2, 2)$, $(2, 2)$, $(2\pm2\sqrt{-2}, 2\mp2\sqrt{-2})$, (∞, ∞).

Fig. 8.

(3) The only example satisfying the third possibility is $x+y=0$, $x^5+x^5=0$. Here, $(x, y)\equiv(0, 0)$, $(0, 0)$, $(0, 0)$, $(0, 0)$, (∞, ∞).

f. $k=1$, $l=6$, and, therefore, $kl=6$. Here, $2n'+n''=6$. There are four possibilities:

n'	n''
3	0
2	2
1	4
0	6

Fig. 9.

The solution of the system $x+y=a_1$, $x^6+y^6=a_6$ leads to a cubic for xy. As the values of x and y are quite complicated, we shall omit them in this case.

(1) If $a_1=3$ and $a_6=65$, $(x, y)\equiv(1, 2)$, $(2, 1)$, (α, β), (β, α), (γ, δ), (δ, γ), the last four sets being imaginary.

Fig. 10.

(2) If $a_1=2$ and $a_6=2$, $(x, y)\equiv(1, 1)$, $(1, 1)$, (α, β), (β, α), (γ, δ), (δ, γ).

(3) The only equation satisfying the third possibility is $x+y=0$, $x^6+y^6=0$.

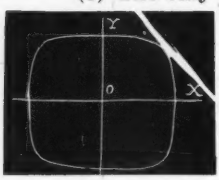

Fig. 11.

(4) The only equations satisfying the fourth possibility are $x+y=0$, $x^6+y^6=0$. There are six solutions each equal to $(x, y)\equiv(0, 0)$.

g. $k=2$, $l=3$, and, therefore, $kl=6$. Here, $2n'+n''=6$.

There are four possibilieies:

n'	n''
3	0
2	2
1	4
0	6

To solve the system $x^2+y^2=a_2$, $x^3+y^3=a_3$, requires the solution of a cubic in xy. If we let $y=rx$, we get the equation,

$$(a_2^3-a_3^2)\left(r^3+\frac{1}{r^3}\right)-3a_3^2\left(r+\frac{1}{r}\right)+2a_2^3=0, \text{ or } (a_2^3-a_3^2)u^3-3a_2^3u+2a_2^3=0,$$

where $u=\left(r+\frac{1}{r}\right)$.

Fig. 12.

(1) If $a_2=13$ and $a_3=35$, $(x, y)\equiv(2, 3)$, $(3, 2)$, $[\frac{1}{2}(2\pm\sqrt{22}), \frac{1}{2}(2\mp\sqrt{22})]$, $[\frac{1}{2}(-7\pm i\sqrt{23}), \frac{1}{2}(-7\mp i\sqrt{23})]$.

(2) In order that the second possibility be satisfied, a_2^3 must equal $2a_3^2$.

If $a_2=8$ and $a_3=16$, $(x, y)\equiv(2, 2)$, $(2, 2)$, $\{\pm[\sqrt{(4\pm2\sqrt{3})}+\sqrt{(\mp2\sqrt{3})}]$, $\pm[\sqrt{(4\pm2\sqrt{3})}-\sqrt{(\mp2\sqrt{3})}]\}$.

(3, 4) In order that the third possibility be satisfied, a_2 and a_3 must both be equal to 0, and these are the only values of a_2 and a_3 which will also satisfy the fourth possibility.

Also solved by G. B. M. Zerr.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

120. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Find the prime numbers p for which $x^2-pxz-px-z+p^2-3=0$ has more than two sets of positive integral solutions x, z, each $<p$.

Solution by H. S. VANDIVER, Bala, Pa.

About the time this problem was originally proposed, Professor Dickson wrote to me that he encountered it in determining those factors of $(p^2-1)^2$ which are of the form $1+px$, p prime. It will first be shown that the latter problem is equivalent to finding all the integers p, x, and z (p a prime) which satisfy

$$x^2-pxz-px-z+p^2-3=0 \ldots (1).$$

Let $(p^2-1)^2=(1+px)(1+py)$, where x and y are positive integers. Expanding and dividing by p,

$$p^3-2p=x+y+pxy.$$

Hence there is necessarily an integer $z>0$, such that $x+y=pz$. By substitution and division by p,

$$z(px+1)=x^2+p^2-2.$$

Setting $z+1$ for z we obtain (1), and the two problems are equivalent.

Solving (1) for p,

$$2p=x(z+1)\pm\sqrt{[x^2(z+3)(z-1)+4(z+3)]}.$$

For this to hold, there is necessarily a positive integer a such that

$$x^2(z+3)(z-1)+4(z+3)=a^2,$$

subject to the condition that $x(z+1)\pm a$ be double a prime.

Putting $z+3=v$, $a^2-vx^2(v-4)=4v$.

Let $v=k^2u$, where u is an integer not divisible by a square other than unity, then

$$u\beta^2-x^2(k^2u-4)=4 \ldots . (2),$$

where $\beta=a/ku$, necessarily an integer.

The preceding analysis shows that (1) is soluble only for cases in which

(2) is soluble. It will suffice then to indicate a method leading to the complete solution of (2).

Regard u and k in (2) as fixed, and consider the relation as the representation of the integer 4 by a quadratic form in β and x, with coefficients u and k^2u-4. Consider the case where the determinant is of the form $k^2u^2-4u>4^2$. Then all the solutions of (2) can be obtained by using the theorem given in Serret's *Cours D'Algebre Supérieure* (Vol. I, p. 80). Suppose, first, that u and k are both odd; then the continued fraction to be considered is:

$$\sqrt{\frac{k^2u-4}{u}}=[k-1;\ 1,\ u'-1,\ 2,\ v'-1,\ 1,\ 2(ku-1),$$
$$1,\ v'-1,\ 2,\ u'-1,\ 1,\ 2(k-1);\ 1,\ \ldots],$$

where $ku=2u'+1$, $k=2v'+1$, $k>4$. (As a check, note that the period is reversible, $2(ku-1)$ being the middle term). If u is divisible by 2 but not by 4, and if k is odd, (2) reduces to the form

$$t\beta^2-x^2(k^2t-2)=2, \text{ where } u=2t,\ t \text{ not congruent to } 0 \pmod 2,$$

and the development is

$$\sqrt{\frac{k^2t-2}{t}}=[k-1;\ 1,\ kt-2,\ 1,\ 2(k-1);\ 1,\ \ldots] \quad (kt>2).$$

If $u=4s$, and k is odd or even, then (2) becomes

$$s\beta^2-x^2(k^2s-1)=1\ldots(3),$$

and the fraction is

$$\sqrt{\frac{k^2s-1}{s}}=[k-1;\ 1,\ 2(ks-1),\ 1,\ 2(k-1);\ 1,\ \ldots] \quad (ks>1).$$

If k is even (2) reduces to the type (3), u arbitrary. For the few numerical values of k and u satisfying $k^2u^2-4u<4^2$, the Gaussian theory of quadratic forms may be used. The other exceptional cases of $k\leq 4$, $kt\leq 2$, $ks=1$, may be disposed of without difficulty. As an example, applying the Serret theorem to (2) for u and k odd, the development of $\sqrt{[(k^2u-4)/u]}$ shows that the third and eleventh complete quotients are the only ones in the period having the denominator 4. Put $k=5$, $u=3$, in (2), then using the convergent immediately preceding the eleventh quotient,

$$\beta/x=(4,\ 1,\ 6,\ 2,\ 1,\ 1,\ 28,\ 1,\ 1,\ 2)=\tfrac{26275}{5401},$$

and we may verify that $3\times 26275^2-71\times 5401^2=4$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

298. Proposed by W. J. GREENSTREET, Marling School, Stroud, England.

Find an approximation to the difference between the sums of n harmonic and n arithmetic means between a and b, when a is very nearly equal to b.

299. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.

The sides of a triangle and the area are in arithmetical progression. Find their values, and show there is only *one* solution in rational integers.

GEOMETRY.

331. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.

The center of two spheres radii r_1, r_2, are at the extremities of a straight line $2a$ on which as a diameter a circle is described. Find a point on the circumference from which the greatest portion of spherical surface is visible.

CALCULUS.

255. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Find the general values of u and v in terms of x, which satisfy the equations $u^2+l^2(du/dx)^2=v^2$, $u^2+m^2(du/dx)^2=v^2+n^2(dv/dx)^2$.

256. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\sum_{x=0}^{x=\infty} \frac{x^{2m}}{1+x^{2n}} = \frac{\pi}{2n\sin\frac{2m+1}{2n}\pi}$, m and n being positive integers of which n is the greater.

257. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

If $A=\int_0^\infty \frac{dx}{\sqrt{x}\,(2a+x)^n}$, $B=\int_0^\infty \frac{y^n dy}{\sqrt{y}\,(a^2+y^2)^n}$, find A/B.

MECHANICS.

215. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Determine the curve in a vertical plane along a chord of which a particle will slide under the force of gravity and the retardation of friction so that it will traverse the whole length of the chord in a time t which is independent of its direction as long as the upper end of the chord remains fixed. Discuss the result.

NOTES AND NEWS.

Professor Heinrich Maschke, of the University of Chicago, died March 1. 1908, after an illness of a few days. He was born in Breslau, Germany, in 1853, and received his education in the universities of Breslau, Berlin, and Goettingen, obtaining his Doctorate at Goettingen in 1880. For ten years he taught in the Guisenstaedische Gymnasium in Berlin, and came to the University of Chicago as Assistant Professor of Mathematics in 1892, was promoted to the associate professorship in 1896, and to the full professorship in 1906. He was widely known both in this country and abroad for his publications in the leading mathematical journals, and was universally beloved by the hundreds of students who have been under his instruction at the University of Chicago. He had been vice-president of the American Mathematical Society and was a member of the Deutsche Mathematiker-Vereinegung. S.

BOOKS.

Scrapbook of Elementary Mathematics. Notes, Recreations, Essays. By William F. White, Ph. D., State Normal School, New Paltz, New York. 12mo. 248 pages, cloth, gilt top. Price, $1.00 net. Chicago: The Open Court Publishing Co.

This book contains a collection of Essays, Recreations, and Notes on Mathematical Topics, presented in a somewhat humorous style, but not in any way sacrificing accuracy of statement. Many interesting topics are discussed. We give a few of the thirty or more subjects: A Few Numerical Curiosities, Algebraic Fallacies, Geometrical Puzzles, The Three Famous Problems of Antiquity, Mathematical Symbols, A Few Surprising Facts in the History of Mathematics, Magic Squares, Quotations on Mathematics, The Golden Age of Mathematics, Alice in the Wonderland of Mathematics, etc. Some topics are perhaps too briefly treated, yet enough is given to arouse the interest of students of mathematics. F.

Graphic Algebra. By Arthur Schultze, Ph. D., Assistant Professor of Mathematics, New York University, Head of the Department of Mathematics, High School of Commerce. 8vo. Cloth, viii + 93 pages. Price, 80 cents net. New York: The Macmillan Co.

This little book contains a fine collection of problems and many valuable suggestions. The graphic solutions of the Quadratic and Cubic Equations, as well as other equations, are well illustrated. F.

4

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XV. APRIL, 1908. NO. 4.

THE GALOIS GROUP OF A RECIPROCAL QUARTIC EQUATION.

By DR. L. E. DICKSON, The University of Chicago.

1. The Galois group of a given algebraic equation, $E=0$, gives complete information as to its solvability by radicals, the number and degree of independent root extractions necessary in case the equation is solvable, and reflects the possible factorizations of E within the chosen domain of rationality, R. The Galois group, G, is characterized by the properties that

(i) Every rational function of the roots which remains (numerically) unaltered by all the substitutions of G equals a quantity in the domain R;

(ii) Every rational function of the roots which equals a quantity in R remains (numerically) unaltered by all the substitutions of G.

Throughout the discussion, the coefficients of every rational function considered are assumed to belong to the domain R.

In determining the group, G, for a given equation, one usually resorts to more or less fortunate guesses as to the rational functions to be employed with success. The present treatment for

$$E \equiv x^4 - ax^3 + bx^2 - ax + 1 = 0 \tag{1}$$

is straightforward and leads in a natural and pleasing way to the criteria desired. While the domain, R, chosen is an arbitrary domain containing a and b, we shall speak of numbers in R as rational, numbers not in R as irrational, and the reader may, if he prefers concreteness, take R to be the domain of all rational (integral or fractional) numbers.

The Galois group for (1) may be any one of ten groups*; necessary and sufficient conditions on a and b for a specified group are determined, and a summary given in § 8.

As the Galois theory is applied only to equations† with distinct roots, we assume that the discriminant Δ of (1) is not zero. By the usual formula, or more simply by § 4, we find that

*The discussion in the MONTHLY, 1904, p. 195, is incomplete in several respects.

†Multiple factors are first to be determined (by the greatest common divisor process, for example) and removed, as may be done in the domain, R.